

Glimpses of the Octonions and Quaternions History and Today's Applications in Quantum Physics (*)

A.Krzysztof Kwaśniewski
the Dissident

- relegated by Białystok University authorities
from the Institute of Computer Sciences
organized by the author
to Faculty of Physics

ul. Lipowa 41, 15 424 Białystok, Poland

e-mail: kwandr@gmail.com, <http://ii.uwb.edu.pl/akk/publ1.htm>

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Abstract

Before we dive into the accessibility stream of nowadays indicatory applications of octonions to computer and other sciences and to quantum physics let us focus for a while on the crucially relevant events for today's revival on interest to nonassociativity. Our reflections keep wandering back to the Brahmagupta-Fibonacci Two-Square Identity and then via the Euler Four-Square Identity up to the Degen-Graves-Cayley Eight-Square Identity. These glimpses of history incline and invite us to re-tell the story on how about one month after quaternions have been carved on the Brougham bridge octonions were discovered by John Thomas Graves (1806-1870), jurist and mathematician - a friend of William Rowan Hamilton (1805-1865). As for today we just mention en passant quaternionic and octonionic quantum mechanics, generalization of Cauchy-Riemann equations for octonions and Triality Principle and G_2 group in spinor language in a descriptive way in

order not to daunt non-specialists. Relation to finite geometries is recalled and the links to the 7Stones of seven sphere , seven "imaginary" octonions' units in out of the Plato's Cave Reality applications are appointed. This way we are welcome back to primary ideas of Heisenberg, Wheeler and other distinguished founders of quantum mechanics and quantum gravity foundations.

Key Words: composition algebras, finite projective geometry, quantum mechanics

AMS Classification Numbers: 17A75, 51E15, 81Q60

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1 Finite geometries and infinite poem

With this essay we add to Martin Huxley's *An infinite Poem* [1] another strophe:

CAYLEY and HAMILTON and John Thomas GRAVES
 those were and are The BRAVES
 since CAYLAY - DICKSON this is the case
 that two quaternions make octaven tool
 then to be used with G_2 group too ...

This note being on infinite subject has also its references grouped into two parts. First part starts from [1] and runs up to [40] - [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] - which might mark this note temporal end as these are sufficiently abundant and indicative. But life goes

on and then follows an endless sequence, starting from [41]. Moreover. The number of references tells you how old I am. Temporally.

I. Galois numbers

We use the standard convention. Out of the vector space $V(n+1, q)$ we derive the geometric structure $PG(n, q)$ and this will be called the projective geometry of dimension n over $GF(q)$ denoting Galois field. The dimension is to meant so that lines have 1 dimension, planes have 2 dimensions, etc. Let now S be the family of all finite dimensional subspaces $V(n, q)$ of an infinite dimensional linear space over finite Galois field $GF(q)$. Consider these finite geometries posets $\langle S, \subseteq \rangle$ and let define the Galois number $G_{n,q}$ to be

$$G_{n,q} = \sum_k^n \binom{n}{k}_q = \mathbf{Galois\ number} = \text{the number of all subspaces of } V(n, q)$$

Then for $\langle S, \subseteq \rangle$ we awake

$$G_{n,q} = |[x, y]|$$

for such $x, y \in S$ that maximal chain joining x with y is of length n .

Galois numbers play the crucial role in the finite geometries poset $\langle S, \subseteq \rangle$ description and applications including such identities as the following one "q-exp squared identity"

$$[exp_q(x)]^2 \equiv \left(\sum_{n \geq 0} \frac{x^n}{(n_q)!} \right)^2 \equiv \sum_{n \geq 0} \frac{G_{n,q} x^n}{(1+q)(1+q+q^2) \dots (1+q+\dots+q^{n-1})}$$

II. Lattices of subspaces and projective spaces

In accordance with $\binom{3}{1}_2 = \binom{3}{2}_2 = 7$ and $G_{3,2} = 1 + 7 + 7 + 1 = 16$ we characterize the adequate finite geometry for $n = 3, q = 2$ by the Hasse diagram of $L(3.2)$ in Fig.1.

The corresponding projective space $P(2.2)$ is then the plane consisting of 7 points [see: non-zero elements - hence designating lines in $V(3.2)$] and 7 projective lines [see: P_1, \dots, P_7 planes - subspaces of $V(3.2)$]. It is depicted by Fig.2.

Fig.2. delineates the smallest possible projective plane $P(2.2)$ known as the Fano plane. In accordance with the Fig.1. it contains seven points and seven lines. These seven points are shown as dots of intersections, while the seven

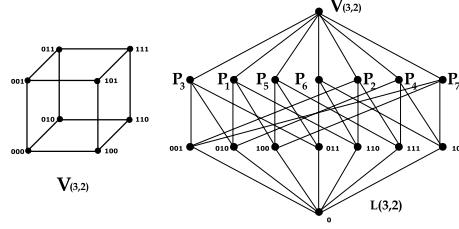


Figure 1: Display of $V(3,2)$ and $L(3,2)$

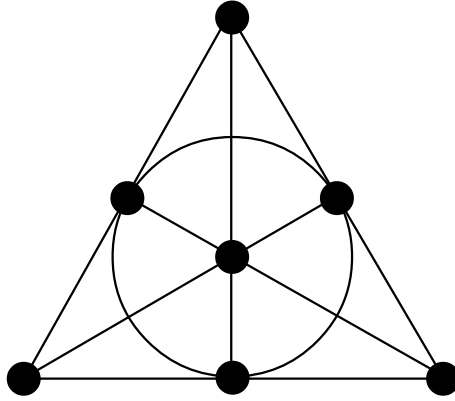


Figure 2: Display of Fano Duality Plane

projective lines are just six line segments and a circle. Note that one could equivalently consider the dots to be the lines while the line segments and circle to be the points. This property is an example of the known duality property of projective planes. This means that if the points and lines are interchanged, the resulting set of objects is again a projective plane. We are dealing with octonions then. See more what follows.

In accordance with $\binom{2}{1}_2 = 3$ and $G_{2,2} = 1 + 3 + 1 = 5$ we characterize the adequate finite geometry for $n = 2, q = 2$ by the Hasse diagram of $L(2,2)$ in Fig.3. where also the [circle] projective line $P(1,2)$ is displayed.

We are dealing with quaternions then. See more what follows. There are many ways to correlate octonions [quaternions] multiplication tables rules with underlying projective geometry picture [9,5,10,15]. Here come quite obvious examples. Using the following adjusted to Fig.1 identification

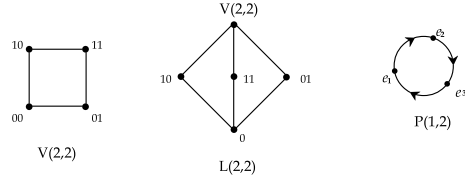


Figure 3: Display of V(2.2), L(2.2) and P(1.2)

Projective lines L_i in $P(2.2)$, $i = 1, 2, 3, 4, 5, 6, 7$, $V(3.2)$ - corresponding
Subspaces-Planes P_i

$$\begin{aligned}
L_1 &= \{100, 110, 010\} \succ P_1 = 000, 100, 110, 010 \\
L_2 &= \{001, 111, 110\} \succ P_2 = \{000, 001, 111, 110\} \\
L_3 &= \{010, 011, 001\} \succ P_3 = \{000, 010, 011, 001\} \\
L_4 &= \{010, 111, 101\} \succ P_4 = \{000, 010, 111, 101\} \\
L_5 &= \{100, 111, 011\} \succ P_5 = \{000, 100, 111, 011\} \\
L_6 &= \{011, 101, 110\} \succ P_6 = \{000, 011, 101, 110\} \\
L_7 &= \{100, 101, 001\} \succ P_7 = \{000, 100, 101, 001\}
\end{aligned}$$

and the following identification of octonion imaginary units

$$e_1 = 010, e_2 = 100, e_3 = 110, e_4 = 001, e_5 = 011, e_6 = 101, e_7 = 111$$

we have

The rules:

$$e_1 e_3 = e_2, e_2 e_6 = e_4, e_4 e_5 = e_1, e_3 e_6 = e_5, e_1 e_7 = e_6, e_2 e_7 = e_5, e_4 e_7 = e_3$$

Accordingly

$$\begin{aligned}
L_1 &= \{e_1, e_2, e_3\}, L_2 = \{e_4, e_7, e_3\}, L_3 = \{e_1, e_5, e_4\}, L_4 = \{e_1, e_7, e_6\}, \\
L_5 &= \{e_2, e_7, e_5\}, L_6 = \{e_5, e_6, e_3\}, L_7 = \{e_2, e_6, e_4\}.
\end{aligned}$$

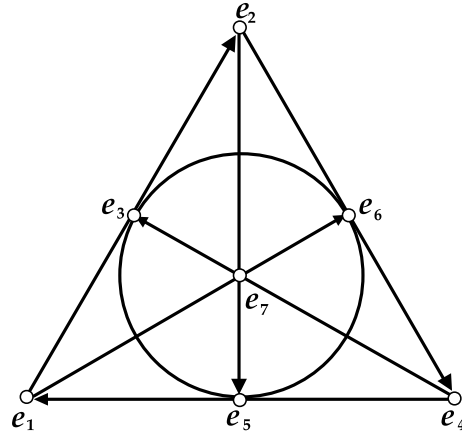


Figure 4: Fig.4 Display of octonion multiplication rules Fano Plane

Using another adjusted to Fig.1 identification of octonion imaginary units

$$e_1 = p_4 = 010, e_2 = p_1 = 100, e_3 = p_2 = 110, e_4 = p_3 = 001,$$

$$e_7 = p_5 = 111, e_5 = p_6 = 011, e_6 = p_7 = 101.$$

we we get the Fano Plane imported from Joseph Malkevitch's "Finite Geometries" [12]; see Fig.5.

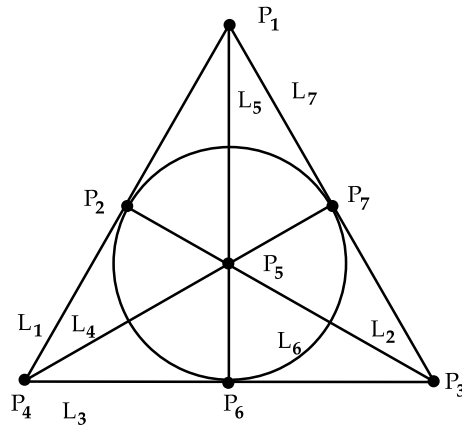


Figure 5: Display of another octonion Fano Plane coding

The still another way is the self-explanatory way of picture rule for octonions multiplication borrowed from Tevian Dray's garner [TD], where the corresponding identifications with preceding presentations are evident. [TD]=Tevian Dray Octonions

<http://www.physics.orst.edu/~tevian/octonions/>
, see more: <http://www.math.oregonstate.edu/~tevian/>

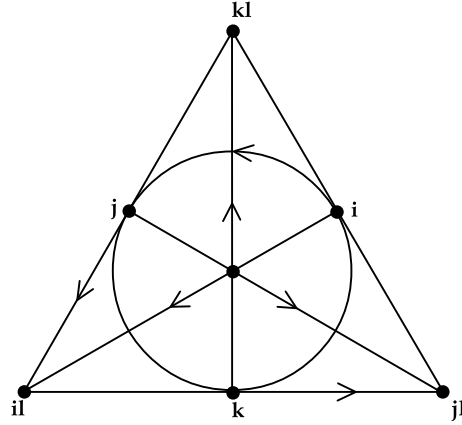


Figure 6: Still another octonion rules coding

Ending this finite geometry glimpses subsection we refer the reader to On-line Notes in Finite Geometry <http://cage.ugent.be/~aoffer/fgw/notes.html> by many distinguished authors and there see Notes on Finite Geometry by Steven H. Cullinane <http://log24.com/notes/>, where the summary Steven H. Cullinane's work in the area of finite geometry during the years 1975 through 2006 is to be found.

III. Octonions, quaternions - wide open access Here now follows afar not complete selective subjectively personal overview description of the nowadays sources **abundant with references** and links to the topics on division algebras in various branches of mathematics and other sciences - intriguing history included such as [2,3,4] and much more deep into the history linking us to the noted Indian mathematician Brahmagupta (598-c.- 665) and Bhaskara Bhaskaracharya II (1114-1185) the lineal successor of Brahmagupta. Bhaskara Bhaskaracharya II also called Bhaskaracarya, or Bhaskara The Learned was the leading mathematician of the 12th century on

our planet. It was Him who wrote the first tractatus with complete and systematic use of the decimal number system. I cannot resist- recalling my first experiences with mathemagics had been reading the Lilavati ("The Beautiful") in Polish by Szczepan Jeleński. The title commemorates and refers to Bhaskara treatise. *Kalejdoskop matematyczny* by Władysław Hugo Dionizy Steinhaus - a reward-gift from him to me in my childhood and then Lilavati those were my first books on the Lilavati Mathemagics; [Mathemagics? see: <http://ii.uwb.edu.pl/akk/>].

Finite geometries. Apart from the reference to finite geometries the book of Professor James Hirschfeld on Projective Geometries over Finite Fields is recommended for the advanced acute and persistent readers. Another source is nice and useful Timothy Peil's Survey of Geometry: <http://www.mnstate.edu/peil/geometry/IndexF/indexold.htm>.

Division algebras: octonions, etc. Here one is welcomed to the cosmic realm of John C. Baez, [<http://math.ucr.edu/home/baez/README.html>] with his unsurpassed "This Week's Finds In Mathematical Physics" and the famous article [5] *The Octonions* from which we quote the Abstract in order to indicate genuine relevance and scope of his ideas and "'weeks'" writings. "The octonions are the largest of the four normed division algebras. While somewhat neglected due to their nonassociativity, they stand at the crossroads of many interesting fields of mathematics. Here we describe them and their relation to Clifford algebras and spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. We also touch upon their applications in quantum logic, special relativity and supersymmetry."

Also the book On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry by John H. Conway and Derek A. Smith [6] is recommended as well as the Geoffrey M. Dixon, book on Division Algebras : Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics [9]. A very important for this essay goals are the articles of of Titus Piezas III and especially this on "The Degen-Graves-Cayley Eight-Square Identity" from The Ramanujan Pages [11]. If you are still in need for short [5 pages! short] Bibliography on quaternions starting with 19-th century - then see John H. Mathews, Bibliography for Quaternions using for example password <http://en.wikipedia.org/wiki/Quaternion>. It starts from *Quaternions* by Christine Ladd in The Analyst, Vol. 4, No. 6. (Nov., 1877), pp. 172-174, and ends with *Passive Attitude Control of Flexible Spacecraft*

from *Quaternion Measurements* by Di Gennaro S. in Journal of Optimization Theory and Applications, January **2003**, vol. **116**, no. 1, pp. 41-60.

For Hamilton's Discovery of Quaternions see excellent paper [19] of the Master B. L. van der Waerden.

Bibliography for octonions? And quaternions? And Clifford algebras? See 94 references in [5] including Jacques Tits papers, Pascual Jordan fundamental papers, Anthony Sudbery papers, V. S. **Varadarajan** book, Ian R. Porteous' book [he showed *it* fresh at the Catenbury 1985 conference devoted to my beloved Clifford algebras], then Feza Grsey and Chia-Hsiung Tze papers [I met? one of them in Paradise [Cocoyoc in 1980]], and first of all - the famous paper by Pascual Jordan, John von Neumann, Eugene Wigner, (**1934**) [I had met no one of them except Wigner] however I had met all of them reading - while studying in Wrocław - with passion *On an algebraic generalization of the quantum mechanical formalism*, (**1934**). Are there any important others' contributions? see also [9,10,15]. And ...there is no end until The End.

It is quite well known how to use the octonions for building all five of the exceptional simple Lie algebras including g_2 intrinsic relevant to parallelizable spheres and triality principle and also interesting for quite abstract immortal theoretical physics [16,17,34]. Here the "magic" use of octonions is crucial if we are not to be overloaded by the much structured respectable and important language of all sophisticated Mathematics for those The Chosen. In *The Octonions Yet Another Essay* by David A. Richter one may find among others the exceptional Lie algebra g_2 treatment with overall ingenious simple use of octonions [<http://homepages.wmich.edu/~drichter/octonions.htm>] as well as the Hamming code and Gosset's lattice and clear realization in octonion geometry picture of an action by the dihedral group D_6 which is the Weyl group for the exceptional 14-dimensional simple Lie algebra g_2 .

As for the quaternion valued functions' analyticity we postpone here any attempt to master the subject of references. Let us however remark on the occasion that the Fueter analyticity is a special case of Clifford analyticity. This is too big theme for glimpses. For octave Cauchy-Riemann equations we refer here to [13,14] as we have no contact with eventual recent developments. Though we shall come back for a while to both quaternion and octonion valued functions (compare with [26,27]) when telling a bit on applications of both in Quantum Mechanical quite recent models or projects in the almost last section with 7Stones aboard hence [7]).

2 John Thomas Graves discovers Cayley numbers before Cayley

Bhaskara in his *Bijaganita* ("Seed Counting"), compiled problems from Brahmagupta and others. He filled many of the gaps in Brahmagupta's work, especially in obtaining a general solution to the **Pell** equation". See <http://www.math.sfu.ca/histmath/India/12thCenturyAD/Bhaskara.html> also for his poetry in mathematical writings. John Pell: 1611 - 1685. Brahmagupta: 598 - 670. It is therefore around 1000 years before Pell when Brahmagupta studied this equation with his Brahmagupta's lemma proved by Brahmagupta in 628 AD by the method called samasa by Indian Mathematicians. see more in Article by: J J O'Connor and E F Robertson ; February 2002, MacTutor History of Mathematics, <http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html> and references therein.

The similar story is about history of what we now call also after Titus Piezas III [11] The Brahmagupta-Fibonacci Two-Square Identity ("just" complex numbers) :

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (bc + ad)^2.$$

At Brahmagupta and then at Fibonacci times this identity was a mystery and enigma - valid for all numbers of those times identity ?! Well- today it is "just" norm composition under product property and Hurwitz theorem but at Brahmagupta Fibonacci times... It is a mystery of our times. The norm of the product is equal to the product of the norms. Why only four? We now know why - anyhow it is a mystery. The Mystery. The Euler Four-Square Identity and the Degen-Graves-Cayley Eight-Square Identity are of the same origin as explained above: "just" norm composition under multiplication property and the Hurwitz's theorem for composition algebras. The **Euler** Four-Square Identity ? Leonhard Euler (1707 - 1783) was born in Switzerland, studied under Johann Bernoulli or Ivan Bernoulli in Petersburg [20]). Sir William Rowan Hamilton (1805 - 1865) had discovered quaternions in 1843 [19]. Hence again - Four-Square Identity was known before quaternions discovery was carved on Broome Bridge in Dublin on October 16 (1843) while William Rowan was being out walking along the Royal Canal with his wife. That is mathematicians' wives destination not only in Ireland.

The **Degen-Graves-Cayley** Eight-Square Identity? First discovered by Fer-



Figure 7: Broome Bridge

dinand Degen (1766-1825) the Danish mathematician around 1818. Hence again - Eight-Square Identity was discovered before octaves.

The Degen- **Graves**-Cayley Eight-Square Identity? It was subsequently independently rediscovered twice: in 1843 by the jurist and mathematician **John Thomas Graves** (1806-1870) and in 1845 by Arthur Cayley (1821-1895). Naturally again the identity follows from the fact that the norm of the product of two octonions is the product of their norms. Octonions were frequently named Cayley numbers - even in [14]. Octaves? Freudenthal's Oktaven? [21] ([21] is reference of [13,14]. I have got a copy of [21] in Russian from the late Professor Ogievietzki at Dubna Institute for Nuclear Research years, years ago.... It was rather top secret!

Do you still remember? - It was Freudenthal who had found out that the five simple Lie groups which are out of the four infinite classical families (exceptional groups and algebras) are related to the isometries of octaval planes [35-37]. For more see [39]; there in P. Kainen's octonion model for physics one encounters many fascinating ideas and vague references. Read and listen (from [39] and ad rem): "In mathematics too, one finds higher-dimensional

objects casting shadows in the lower dimensions". Plato's cave? "For example, the Penrose non-periodic tiling of the plane is a projection of something in dimension at least 4 (Katz) and the Hardy-Ramanujan formula (Andrews) shows that the number of partitions of a positive integer

n may be expressed in terms of the $24k$ -th roots of unity. While integers are 0-dimensional, the partition formula suggests that a 24-fold symmetry is involved - as would be the case if it involves objects in 4-dimensions.”’

Objects casting shadows in the lower dimensions?

This Plato’s cave idea of shadows is everywhere present in an **vedic-minded** John Archibald Wheeler’s writings . Note his famous device motto ”**It from Bit**”’. Wheeler wrote: ”I like to think that someone will trace how the deepest thinking of **India** made its way to **Greece** and from there to the philosophy of our times.”

”‘It is curious that people like Schroedinger, Niels Bohr, Oppenheimer and **John Wheeler** are **Upanishad** scholars.”’ This was another glimpse from the article - Indian Conquests of the Mind - By **Saibal Gupta**. The Statesman.org). The next glimpse is: ”One has the feeling that the thinkers of the East knew it all, and if we could only translate their answers into our language we would have the answers to all our questions.” (source: *Uncommon Wisdom* - By Fritjof Capra p. 40).

Werner Heisenberg should be perhaps also counted an Upanishad scholar. For sure he was neoplatonic in thinking and feeling Quantum Mechanics message. Werner Karl Heisenberg (1901-1976) German theoretical physicist was one of the leading scientists of the 20th century. Heisenberg spent some time in India as **Rabindranath Tagore**’s guest in 1929. There he got acquainted with Indian philosophy which brought him great comfort for its similarity to modern physics. Anyhow is it not like that with all of them: Wheeler, J. A. and Zurek, W. H. and Schroedinger, Niels Bohr, Oppenheimer and you and me - all we are **It from Bit**? This was a kind of passionate idea of Feynman at the dusk of his life.

See *Simulating physics with computers* by Feynman, IJTP 21 1982 [contact ! <http://www.stardrive.org/Jack/Feynman.html>], See Tony Hey memorial http://www.quniverse.sk/buzek/zaujimave/p257_s.pdf. ... **It from Bit**?

Ad **It form Bit** and my private Quantum Plato’s cave use Google password **John Archibald Wheeler to Tachion** and then once you are on Home Page AKK . There in **publikacje** <http://ii.uwb.edu.pl/akk/publ1.htm> see and/or download the reference [38] here and ref. [38] there. If you click the link **John Archibald Wheeler to Tachion** right at the beginnig of my home page then you will see something that you have never seen before , something written by John Archibald Wheeler to Tachion i.e. to me. As now I am the local dissident so I have got a resistance nick-name : Tachion

i.e. a person who causes authorities' contraventions in their past reacting on these from their future.

Let us however continue our main story. Our story without authorities.

Octaves? This is a name given to octonions by Hamilton. This story and the story on how octonions were discovered by John Thomas Graves is quite well known. Perhaps the most complete available knowledge is John Baez's week152 [8] and links there and [19] masterpiece. Therefore instead of repeating the unsurpassed in naturality and reliability John Baez's story we add something perhaps not that well known to the audience curious in the circumstances we are now involved with interest in. Let me indicate the presence of John Thomas Graves' **brothers** in his and Sir William Rowan Hamilton's quaternion-octaven life story. Let me tell you something about **Charles Graves** Born: 12 November (or 6 December) 1812 and Died: 1899. Sir William Rowan Hamilton died in 1865. The Memorial Address was delivered by Charles Graves, who was President of the Royal Irish Academy at the time of Hamilton's death

<http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Eloge/Eloge.html>

His loge was delivered at the Stated Meeting of the Royal Irish Academy on the 30th November, 1865. The Graves brothers had known Hamilton from his college days: the elder brother John Graves had been a classmate of Hamilton. Charles Graves had been a friend and a colleague of Hamilton, as Professor of Mathematics at Trinity 1829-1835 Trinity College in Dublin and at the same time was a brother of Robert Perceval Graves who was Hamilton's first biographer.

Charles Graves was elevated up to the position of Bishop of Limerick. He was a noted linguist and antiquarian, and was the grandfather of the poet Robert Graves. This is known quite well. The less known fact [I have learned about it from Professor O.V. Viskov from Steklov Institute] - the less known fact is that the Bishop of Limerick - Charles Graves in [22] applied what is now sometimes called the Blissard umbral calculus and now also referred to Rota the operator algebra. This algebra had been practiced already in 19-th century, then the algebra rediscovered in quantum physics as Heisenberg-Weyl algebra in 20-th century [23,24]. This is crucial algebra of Quantum Theories including lasers. The present author in a bunch of research papers had introduced the name **Graves Heisenberg Weyl Algebra** - GHW Algebra in short, for this nowadays exploited algebra in plenty of applications. The

name GHW algebra is already known to and has been e-mail discussed with the distinguished Professor Patric D.F. Ion!

For GHW very first references and sometimes the papers to be loaded it is enough to write into Google the password: *Graves Heisenberg Weyl Algebra*. One is referred also to my another **Indian paper** in Bulletin of the Allahabad Mathematical Society paper from **2005** [25]. For GHW algebra setting and references see also an **Indian paper** [20]. Finally - one may contact "publikacje" on <http://ii.uwb.edu.pl/akk/publ1.htm> .

3 Snapshots on spheres to comb and even parallelizable ones , seven "imaginary" octonions' units in out of the Plato's Cave Reality quantum applications and 7Stones

This is a section that should be written. Or rather it should be a book of books telling the nowadays story of octonions' life in Mathematics, Gravity and Quantum theories and invisibly playing a role in everyday's life of seven stones from the Cyber Space password <http://www.7stones.com/>.

Here I am going to share with you some personal, hence partly accidental expectations what should be there in this book of books. References quoted are by no means representative to the subject being defined on the way anyhow. Here come snapshots. Just like a running view through the opened window of your running car. The weather is all right. Snapshots.

The Cauchy integral formulas and Cauchy - Riemann Equations? For octonions? - Yes. For example [26,27,...?,13,14] and universal reference [5] with 94 references with other references therein.

Aspects of Quaternion and or Octonion Quantum Mechanics? - Yes. For example [28-33] and universal reference source[5] with 94 references with other references therein and [8] with week192. A contact with Paul C. Kainen's *An Octonion Model for Physics* **2000** is recommended too [39]. The note [30] is referring to Octonionic Gauge Theory and the letter [31] makes reference to seven dimensional octonionic Yang-Mills instantons and string solitons. A few months years old young and small note [32] brings us news about algorithms for least squares problem in quaternionic quantum theory.

Quaternionic Quantum Mechanics and Quantum Fields ? Yes and see why not?! See [33]. It is the 608 pages book from year 1995. Here we quote an abstract of its subjects' content:

"‘It has been known since the 1930s that quantum mechanics can be formulated in quaternionic as well as complex Hilbert space. But systematic work on the quaternionic extension of standard quantum mechanics has scarcely begun. ... book signals a major conceptual advance and gives a detailed development and exposition of quaternionic quantum mechanics for the purpose of determining whether quaternionic Hilbert space is the appropriate arena for the long sought-after unification of the standard model forces with gravitation. Significant results from earlier literature, together with many new results obtained by the author, are integrated to give a coherent picture of the subject. The book also provides an introduction to the problem of formulating quantum field theories in quaternionic Hilbert space. The book concludes with a chapter devoted to discussions on where quaternionic quantum mechanics may fit into the physics of unification, experimental and measurement theory issues, and the many open questions that still challenge the field.’"

For those necessitating in further motive encouragement another quotation [from Girish Joshi's MATHEMATICAL STRUCTURES IN NATURE, Beyond Complex Numbers; just a glimpses:

<http://www.ph.unimelb.edu.au/~ywong/poster/articles/joshi.html>

"‘IN NATURE, there is a deep connection between exceptional mathematical structures and the laws of micro- and macro-physics — Quaternions and Octonions have played an important role in the recent development of pure and applied physics.

Quaternions were discovered by Hamilton in 1843 [19] and the Quaternions' main use in the 19th century consisted in expressing physical theories in "Quaternionic notation". An important work where this was done was Maxwell's treatise on electricity and magnetism. Toward the end of the century, the value of their use in electromagnetic theories led to a heated debate dubbed "The Great Quaternionic War".

In a 1936 paper, Birkhoff and von Neumann presented a propositional calculus for Quantum Mechanics and showed that a concrete realization leads to the general result that a Quantum Mechanical system may be represented as

a vector space over the Real, Complex, and Quaternionic fields. Since then this area has remained active, aiming to extend Complex Quantum Mechanics (CQM) by generalizing the complex unit in CQM to Quaternions and to find observable effects of QQM. Jordan Algebras were proposed by Jordan, Neumann and Wigner in formulating non-associative Quantum Mechanics, where quantization is achieved through associators rather than commutators. This formulation allows mixing of space-time and internal symmetries. Another attractive feature of Jordan Algebra is that critical dimensions of 10 and 26 arise naturally, suggesting a connection to string theory.

Away from physics, Quaternions have recently been used for robotic control, computer graphics, vision theory, spacecraft orientation and geophysics. The space shuttle's flight software uses Quaternions in its guidance navigation and flight control computations.

”‘Octonians’”. ”‘In the early days of Quantum Mechanics, there were several attempts to introduce new algebraic structures in physics. In recent years, there has been remarkable activity in the field of physical applications of non-associative algebras: Octonian formulations of **Yang**-Mills gauge theories and string theory, Octonian description of quarks and leptons, and Octonian supergravity.’”

At that hight of generality of a posteriori inclination Girish Joshi is perfectly right on the right side of the more and more common convictions.

The Practical use of division algebras in gravity and cosmology Theories is not a new invention at all! For example see the 1971 year paper [34]. For an Online Article posted on **2007-05-26** 20:16:11 abundant with ideas and links to highly respective Mathemagics see [39].

More on QM and octonions links - another snapshots

Geoffrey Dixon Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics is at hand [9]. What is it about? Abstract: ”‘The four real division algebras (reals, complexes, quaternions and octonions) are the most obvious signposts to a rich and intricate realm of select and beautiful mathematical structures. Using the new tool of adjoint division algebras, with respect to which the division algebras themselves appear in the role of spinor spaces, some of these structures are developed, including parallelizable spheres, exceptional Lie groups, and triality. In the case of triality the use of adjoint octonions greatly simplifies its investigation. Motivating this work, however, is a strong conviction that the design of our

physical reality arises from this select mathematical realm. A compelling case for that conviction is presented, a derivation of the standard model of leptons and quarks.”’

Susumo Okubo, Introduction to Octonion and Other Non-Associative Algebras in Physics. What is it about? It covers topics ranging from algebras of observables in quantum mechanics and octonions to division algebra, triple-linear products and YangBaxter equation. Here you may find the non-associative **gauge** theoretic reformulation of general relativity theory. [You want to see present Tachion Kwaśniewski as a student talking with **Yang**? Chen Ning Yang? [**Yang-Mills gauge** theory] Contact my website <http://ii.uwb.edu.pl/akk/index.html> . In 1976 the present author was young and was invited by Yang to SUNY - the State Univesity of New York - for one semester. When I was ten -in 1957 Tsung Dao Lee and Chen Ning Yang won the Nobel Prize in physics.

Murray Bremner Quantum Octonions [29]. What is it about? Abstract: Bremner constructs a quantum deformation of the complex Cayley algebra - a kind of q-deformed version of the octonions. He uses the representation of $U_q(sl(2))$, the quantized enveloping algebra of the complex Lie algebra $sl(2)$.

Murat Gunaydin, Hermann Nicolai,[31]; write on **seven dimensional** octonionic Yang-Mills instanton. What is it about? Abstract: They construct an octonionic instanton solution to the **seven** dimensional Yang-Mills theory based on the exceptional gauge group G_2 which is the automorphism group of this extreme division algebra of octonions. This octonionic instanton has an extension to a solitonic solution of the low energy effective theory of the heterotic string that preserves two of the sixteen supersymmetries and hence corresponds to $N = 1$ space-time supersymmetry in (2+1) dimensions transverse to the **seven dimensions** where the **Yang-Mills** instanton is defined.

...

Seven Stones snapshot. The Author: Seven Stones Multimedia [7]. What is this about? This is a series of tutorial modules which apart from various vivid labs includes such topics as lessons about ”‘the **octonion algebra**, division algebras, lie algebras, lie groups, and spinors (Clifford algebras); introduction to special relativity; introduction to quantum mechanics, quantum mechanical uncertainty, nuclear decay and halflife lab, Bohr atom lab; statistics (introduction to linear regression); and astronomy (expanding universe, Doppler shift).”’

SO(8), Triality, F4, and seven sphere parallelizability ? . [One day one had seen there in the sky over Poland a flying exceptional structure **F16**] with a new minister on board. This was not a Lie algebra, not at all. Well, revenons a nos moutons! Where you might be served the tutorial answer? Naturally in [7] i.e. in **Seven Stones** Multimedia. Contact <http://www.7stones.com/Homepage/octotut14.html> i.e. the lesson 14 and others at wish. You might have a look also on parallelizability and triality writings [16,18] supposed to be concerned with physics theoretically. For more recent paper - see [18]. Naturally J. Baez is as always around and helpful [5,8].

The seven sphere parallelizability final history was born in *Częstochowa* - the Holy Place of Catholics of Poland and all over the Globe being so much praised by our Santo subito John Paul II. [Recall Castel Gandolfo meetings with Him - devoted also to Mathemagics.]

The final parallelizability history was born there in *Częstochowa* as far as the main contributor's birth is concerned.

You know, the classical problem was to determine which of the spheres S^n are parallelizable. For $n = 1$ this is the circle in the Gauss plane. For $n = 3$ the 3-sphere which is also the group $SU(2)$ and the only other parallelizable sphere is S^7 . This was proved in 1958, by Michel André Kervaire, and then by Raoul Bott and John Milnor, in independent work. note then [see WIKI] he died on month ago: Michel André Kervaire (*Częstochowa*, Poland, 26 April 1927 - Geneva, Switzerland, **19 November 2007**)

He worked for some time with **Milnor** [from SUNY - the State University of New York; Professor Yang was there too]. As **seven** is concerned "...John Milnor...-his most celebrated single result is his proof of the existence of **7-dimensional spheres** with nonstandard differential structure. Later with Michel Kervaire, he showed that the **7-sphere** has 15 differentiable structures (28 if you consider orientation). An n -sphere with nonstandard differential structure is called an exotic sphere, a term coined by Milnor."

see <http://mathworld.wolfram.com/Parallelizable.html>.

Do not be afraid of nonassociativity of octonions. These are almost associative as forming an example of the alternative algebra [13,14]. No surprise then that in [40] H. Albuquerque and S. Majid, observed that the "octonions should have just the right properties as a substrate for the study of subtle properties of associativity in tensor product algebras. It is well-known that octaves form an *alternative* algebra; associativity holds when two of the three

terms are equal; see, e.g., Schafer . More; ”‘Albuquerque and Majid have proved that the octonions are associative up to a natural transformation’’ - see also my invention in [13,14]. ”‘Their work uses ideas from quantum algebra and from Mac Lane’s theory of coherence in categories. The introduction of a thoroughly octaval viewpoint into the *topos* itself ought to have very interesting consequences for the enterprise of building a categorical model of continuum mechanics’’... [39].

4 END

There is no end.

4.1. Example of no end.

Added in proof 1. After many whiles of good Google safe sale browsing in the Internet Cosmos, after more readings and after receiving letters it is of outmost importance to indicate at least the following positions symptomatic for the subject of our interest here. This concerns supersymmetry and the division algebras [41] and integrable hierarchies [42]. We were also encouraged and inspired to look through interesting papers of Vladimir D. Dzhunushaliev thus having been led to articles by Merab Gogberashvili [43,44]. The author of [43] uses the algebra of split octonions gaining an original octonionic version of Dirac Equations. The natural appearance of octonionic gradient gives us a hint to use the octonionic analyticity as introduced by the author in [13,14]. The deeper gain seems to be achieved in the next paper [44] on octonionic version of electrodynamics while noting that the non-existence of magnetic monopoles in classical electrodynamics is related to an associativity limit of Octonionic electrodynamics from [44].

Vladimir D. Dzhunushaliev works [45-49] end up with ”‘Toy Models of a Nonassociative Quantum Mechanics’’[47] and supersymmetric extension of quantum mechanics via introducing octonionios modulo quaternions! as ”‘hidden variables’’. Both Merab Gogberashvili papers [43,44] as well as A. Dimakis and F. Muller-Hoissen [2] clue reference are referred to by Vladimir D. Dzhunushaliev in his papers from our list of his references.

In order not to get an impression that OQM [Octonionic Quantum Mechanics] is a new subject let us at last but not least recall as an example from the past century the important paper [50] from 1996 by Stefano De Leo and Khaled Abdel-Khalek in which the authors solve the hermiticity problem and

define an appropriate momentum operator within OQM. For more - including also Octonionic Dirac Equation's matters see: <http://ptp.ipap.jp/cgi-bin/getarticle?magazine=PTP.....page=833-845> There you find also substantial references to early papers by Professors L. P. Horwitz and L. C. Biedenharn, [51] and Professor Jakub Rembielinski from Poland [52]. As a matter of fact the dissertation of Professor Jakub Rembielinski was my first contact with OQM. Of course apart from fundamental [no way out?] paper from 1934 by Jordan , von Neumann and Wigner [53] which I had gone through as a student <http://ii.uwb.edu.pl/akk/>.

4.2. Example of no end.

The recent references from People's Republic of China on quaternion's applications that follow those from the **4.1. Example of no end.** are the effect of e-mail correspondence.

4.3. Example of no end. The Reader is welcomed to contribute. The Example is opened.

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<http://math.uwb.edu.pl/zaf/kwasniewski/index.html> and Philosopher of Art and Eternal Life Kosma-Damian. And the last but not least thanks to my wife for being so withstanding person during the past and forthcoming life keeping calm in the midst of the storm of mathemagics and philosophy of her sons and husband.

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